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Phase space approach to quantum dynamics

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Abstract. We replace the Schrödinger equation for the time propagation of states of a quantized 2D spherical phase space by the dynamics of a system of N particles lying in phase space. This is done through factorization formulae of analytic function theory arising in coherent-state representation, the 'particles' being the zeroes of the quantum state. For linear Hamiltonians, like a spin in a uniform magnetic field, the motion of the particles is classical. However, nonlinear terms induce interactions between the particles. Their time propagation is studied and we show that, contrary to integrable systems, for chaotic maps they tend to fill, as their classical counterpart, the whole phase space in a uniform way.

1. Introduction

In classical mechanics, when an integrable system is perturbed some invariant tori are broken and replaced by small layers where chaotic motion takes place. This process is amplified as the perturbation increases, the way this transition occurs being described by the KAM theorem. We do not have an equivalent quantum mechanical description, and our understanding of the mathematical and physical implications of the system being classically chaotic is still rudimentary [1].

Recently, Lebœuf and Voros [2] proposed a representation of quantum states particularly well adapted to semiclassical analysis. For compact phase spaces such as the 2D torus or the sphere $\mathcal{S}^2$, wavefunctions in coherent-state representation exhibit a finite number N of zeros in phase space, and their location completely determines the state of the system. Moreover, they observed that the distribution of zeros for eigenstates of quantized systems reflects, in the semiclassical regime $\hbar \to \infty$, the nature of the underlying classical dynamics: it is 1D for eigenstates of integrable systems while it tends to spread all over phase space in the case of a classically chaotic dynamics.

The purpose of this paper is to develop this representation, and consider its dynamical aspects. Since the zeros completely determine the state of the system, we can interpret quantum mechanics as a (classical) system of N particles—the zeros—lying in phase space. The nature of their interaction and the presence of external fields acting on them will depend on the Hamiltonian defining the system. A stationary state will be associated, in this picture, with an 'equilibrium' configuration of the system of particles, and there will be as many of them as there are stationary states.

However, an arbitrary initial configuration of zeros will in general evolve in time, in the same way that an arbitrary initial state $|\psi(0)\rangle$ will evolve in time according to the Schrödinger equation. In section 3 we rewrite the Schrödinger equation in this new

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picture, and determine the equations of motion governing the time propagation of the zeros. In doing that, we will clarify the nature of their interaction and the possible presence of external fields. For simplicity, we will restrict ourselves to the case of Hamiltonians written in terms of the generators of the SU(2) group—a spin system—for which phase space is a 2D sphere and the analytic coherent-state representation of Hilbert space consists of ordinary polynomials of degree $2s$. Some preliminary notions are introduced in section 2. Finally, in section 4 we illustrate the results considering two examples.

2. The SU(2) coherent-state representation

We recall first some basic formulae arising in the context of SU(2) coherent-state representation. Consider a Hamiltonian $H = H(S_z, S_+, S_-, t)$ written in terms of the generators of the SU(2) group, satisfying the algebra

$$[S_z ; S_\pm] = \pm \hbar S_\pm \qquad [S_- ; S_+] = -2\hbar S_z. \tag{2.1}$$

Since H commutes with $\boldsymbol{S}^2$, the Hilbert space $\mathscr{H}$ is $(2s+1)$-dimensional and spanned by the usual discrete basis

$$S_z|m\rangle = m\hbar|m\rangle \qquad m=-s,\ldots,s \qquad \boldsymbol{S}^2|m\rangle = s(s+1)\hbar^2|m\rangle \tag{2.2}$$

while the phase space $\mathscr{P}$ is a 2D sphere (the Riemann sphere). We henceforth take

$$\hbar = \frac{1}{[s(s+1)]^{1/2}} \tag{2.3}$$

so that the radius of the sphere is fixed to 1. With this convention, the phase space can then be labelled by polar and azimuthal angles (θ, ϕ). Moreover, the semiclassical limit now corresponds to $s \to \infty$.

An alternative representation of $\mathscr{H}$ in terms of analytic functions is obtained through the spin coherent states [3]

$$|z\rangle = \mathrm{e}^{zS_+/\hbar}|-s\rangle \qquad z = \cot\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\phi} \tag{2.4}$$

having the norm $\langle z|z\rangle = (1+z\bar{z})^{2s}$. The complex variable z spanning $\mathscr{P}$ corresponds to a stereographic projection of the Riemann sphere onto the plane by the north pole. The state $|z\rangle$ is a minimum-uncertainty packet on $\mathscr{P}$ centred at $z=(\theta,\phi)$ with a width $\mathscr{O}(\hbar^{1/2})$ in both directions, thus coherent states are the most 'classical' quantum states of angular momentum. They have the property

$$\langle z|m\rangle = \binom{2s}{s+m}^{1/2} z^{s+m} \tag{2.5}$$

so that the coherent-state decomposition of a state $|\psi\rangle = \Sigma_{-s}^{s}\, a_m|m\rangle$ of $\mathscr{H}$

$$\psi(z) = \langle z|\psi\rangle = \sum_{m=-s}^{s} c_m z^{s+m} \qquad c_m = \binom{2s}{s+m}^{1/2} a_m \tag{2.6}$$

is a polynomial of degree $2s$. It therefore has $N=2s$ complex zeros $\{z_k\}_{k=1,\dots,N}$ on the sphere. We now exploit this analytic structure: the standard factorization formula

$$\psi(z)=\mathcal{N}\prod_{k=1}^{N}(z-z_k) \tag{2.7}$$

where $\mathcal{N}$ is a normalization constant, provides a one-to-one mapping between states $|\psi\rangle$ of $\mathcal{H}$ and configurations $\{z_k\}_{k=1,\dots,N}$ of zeros in $\mathcal{P}$. The spin quantum state is therefore uniquely determined (up to a constant factor) by a set $\{z_k\}$ of $N=2s$ points on $\mathcal{P}$. This representation was already known to Majorana [4] in 1932. As he suggested, we may associate with each zero z_k corresponding to a point P on the Riemann sphere a unit vector $\boldsymbol{OP}$, O being the centre of the sphere. The quantum state of the spin s particle (which classically corresponds to a single arrow pointing in a definite direction) is then 'composed' of a set of $2s$ unit vectors (spins) pointing in arbitrary directions $\{\boldsymbol{OP}_k\}_{k=1,\dots,N}$. As we show below, a coherent state is a very particular state for which the $2s$ spins point in the *same* direction.

In spite of being particularly simple for spin systems, a similar parametrization by the zeros can be applied to other spaces of entire functions; in particular, this has been done in [2] for the case of the 2D toric phase space. Moreover, an analogue factorization structure has also been exploited in the context of the quantum Hall effect [5, 6] and also in 2D electrons in a periodic magnetic field [7].

Some relevant features of this *exact* parametrization are [2]:

(i) the *scalar* product of two arbitrary states $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$ defined, respectively, by the set of zeros $\{z_{\alpha k}\}_{k=1,\dots,N}$, $\{z_{\beta k}\}_{k=1,\dots,N}$ is given by

$$\langle\psi_\alpha|\psi_\beta\rangle=\mathcal{N}_\alpha\mathcal{N}_\beta\sum_{m=0}^{N}\bar{g}_{\alpha m}g_{\beta m} \tag{2.8}$$

where

$$g_{\gamma m}=\sum_{1=k_1<k_2<\dots<k_m}^{N}\binom{N}{m}^{1/2}z_{\gamma k_1}z_{\gamma k_2}\dots z_{\gamma k_m}. \tag{2.9}$$

Equations (2.8) and (2.9) follow from the standard connection between the coefficients of a polynomial and its zeros. In terms of the latter, the normalization constant $\mathcal{N}$ of an arbitrary state $|\psi\rangle$ (cf (2.7)) is thus expressed as

$$\mathcal{N}=\left[\sum_{m=0}^{N}|g_m|^2\right]^{-1/2}.$$

(ii) The number $N=2s$ of zeros is controlled by the value of $\hbar$ (or vice versa, cf (2.3)), and tends to infinity in the semiclassical limit (i.e. a thermodynamic limit).

(iii) Each zero constitutes a topological defect of $\psi(z)$, since doing a small closed path around a (non-degenerate) zero changes its phase by 2π.

(iv) The Husimi distribution

$$W_\psi(z,\bar{z})=\frac{|\psi(z)|^2}{\langle z|z\rangle}=\frac{|\mathcal{N}\Pi_{k=1}^{N}(z-z_k)|^2}{(1+z\bar{z})^{2s}} \tag{2.10}$$

has the same zeros as $\psi(z)$. Thus, through its zeros, this positive-definite real function defined on $\mathcal{P}$ carries the full quantum information of the state of the system.

Since $\langle z|z_0\rangle=(1+\bar{z}_0z)^{2s}$ [3], a wavepacket $|z_0\rangle$ has a particularly simple representation: it is defined by placing the $2s$ zeros at the single point $z=-z_0/|z_0|^2$ in $\mathcal{P}$, the opposite point to z_0 on the sphere.

Finally, in this representation, the generators (2.1) have the following realization by differential operators:

$$\begin{aligned} \langle z|S_z|\psi\rangle &= \hbar(z\partial_z - s)\psi(z) \\ \langle z|S_+|\psi\rangle &= \hbar z(2s - z\partial_z)\psi(z) \\ \langle z|S_-|\psi\rangle &= \hbar\partial_z\psi(z). \end{aligned} \tag{2.11}$$

3. The equations of motion

Our purpose now is to replace the Schrödinger equation $i\hbar\partial_t|\psi\rangle = H|\psi\rangle$ for the time evolution of a state $|\psi(t)\rangle$ of $\mathscr{H}$ by an *equivalent* equation for the time evolution of the zeros.

The coherent-state representation of the Schrödinger equation can be written

$$i\hbar\partial_t\psi(z,t) = \hat{H}(z,\partial_z)\psi(z,t). \tag{3.1}$$

Given the Hamiltonian H as a function of the generators S_z, S_+, S_-, the symbol $\hat{H}(z,\partial_z)$ defined as $\langle z|H|\psi\rangle = \hat{H}(z,\partial_z)\psi(z)$ is easily obtained using (2.11). We will assume that it has been *normal ordered*, i.e. all the ∂_z have been placed to the right of the z variables. If at a time t the state has a zero at a certain phase space point $z = z_k$, then the condition $\psi(z_k + \delta z_k, t + \delta t) = 0$ implies

$$\dot{z}_k = \frac{\delta z_k}{\delta t} = -\left.\frac{\partial_t\psi}{\partial z\psi}\right|_{z_k(t)} \qquad k = 1, \ldots, N. \tag{3.2}$$

Using (3.1) we get

$$\dot{z}_k = \frac{i}{\hbar}\left.\frac{\hat{H}(z,\partial_z)\psi(z,t)}{\partial_z\psi(z)}\right|_{z_k(t)} \qquad k = 1, \ldots, N \tag{3.3}$$

where the function computed on the right-hand side must be evaluated at $z = z_k(t)$, the position of the zero at time t. These equations determine the dynamics of each of the N zeros. However, it is not *fully* written in terms of them, since we still use the function $\psi(z)$ to calculate $\dot{z}_k$.

Let us now assume a polynomial expansion for the symbol $\hat{H}(z,\partial_z)$

$$\hat{H}(z,\partial_z) = \sum_n f_n(z)\partial_z^n \tag{3.4}$$

where ∂_z^n means the nth derivative with respect to z and $f_n(z)$ are arbitrary functions depending on the functional form of H. The higher meaningful order n in (3.4) is $n^* = 2s$ since $\partial_z^n\psi(z) = 0$ for $n > n^*$. The operator ∂_z can be interpreted as a destruction of one zero since it lowers the degree of the polynomial by one (further evaluation of the function at $z = z_k$ annihilates the kth zero). Conversely, multiplication by $(z - z_0)$ creates a zero at $z = z_0$. Then

$$\left.\frac{\hat{H}(z,\partial_z)\psi(z,t)}{\partial_z\psi(z)}\right|_{z_k} = \sum_n f_n(z_k)\left.\frac{\partial_z^n\psi}{\partial_z\psi}\right|_{z_k}. \tag{3.5}$$

But from the factorization formula (2.7)

$$\left.\frac{\partial_z^n\psi}{\partial_z\psi}\right|_{z_k} = \sum_{\substack{1=k_1<k_2<\ldots<k_{n-1} \\ (\neq k)}}^{N} \frac{n!}{(z_k - z_{k_1})(z_k - z_{k_2})\ldots(z_k - z_{k_{n-1}})} \tag{3.6}$$

we obtain a formula allowing the elimination of $\psi(z)$ in (3.3). Each term on the right-hand side of (3.6) can be interpreted as an n-body interaction between the zeros. For an order n, the kth zero interacts simultaneously with $n-1$ of the remaining $N-1$ zeros. There are

$$\binom{N-1}{N-n}$$

different possibilities to choose the $n-1$ zeros, and all these possibilities are taken into account by the sum in (3.6). Then the coefficient $f_n(z_k)$ in (3.5) can be seen as a position-dependent 'charge'. However, this charge depends only on the position of the kth zero, and not on the position of the remaining zeros. This asymmetry reflects the fact that in general (3.3) cannot be written as a derivative of a potential $\dot{z}_k = \partial_{z_k} V(z_1, \ldots, z_N)$ since $\partial_{z_j}\dot{z}_k \neq \partial_{z_k}\dot{z}_j$.

Whether or not n-body interactions exist among the zeros is determined by the operator (3.4), which in turn is fixed by the functional dependence of the Hamiltonian on the generators of the group. If this dependence is simply *linear* (i.e. a spin in a uniform magnetic field), then (2.11) and (3.6) show that only $n=1$ terms would appear, and there will be no interactions between the zeros. However, the coefficient $f_1(z_k)$ in (3.5) will provide an 'external field', and the motion of a given zero will depend only on its own position.

The simplest non-trivial case are quadratic terms on the generators, implying two-body interactions

$$\left.\frac{\partial_z^2\psi}{\partial_z\psi}\right|_{z_k} = 2\sum_{j(\neq k)=1}^{N} \frac{1}{z_k - z_j}. \tag{3.7}$$

This kind of term, as the general n-body interaction (3.6), produces strong short-distance correlations among the zeros. They are responsible in nonlinear Hamiltonians for the spreading of a wavepacket, since, as we already pointed out, a wavepacket $|z_0\rangle = |\theta_0, \phi_0\rangle$ centred at (θ_0, ϕ_0) on $\mathscr{P}$ is defined by placing all the zeros at a single point, the opposite one to z_0 on the sphere. For that configuration, interaction terms like (3.7) are singular and play a dominant role in the short-time dynamics.

The final form for the equations of motion is

$$\dot{z}_k = \frac{\mathrm{i}}{\hbar}\sum_n f_n(z_k) \sum_{\substack{1=k_1<k_2<\ldots<k_{n-1} \\ (\neq k)}}^{N} \frac{n!}{(z_k - z_{k_1})(z_k - z_{k_2})\ldots(z_k - z_{k_{n-1}})} \qquad k = 1, \ldots, N. \tag{3.8}$$

The equilibrium configurations for the zeros—the eigenstates—will be determined by the conditions

$$\dot{z}_k = 0 \qquad k = 1, \ldots, N. \tag{3.9}$$

This will be in general an algebraic system of $N = 2s$ nonlinear coupled equations for the $2s$ variables $\{z_k\}$.

4. Two examples

We now consider two specific examples. If the Hamiltonian H does not depend on time, then we have a 1D integrable system. In order to have a classically chaotic dynamics, we must consider time-dependent Hamiltonians. The simplest case is a periodic sequence of delta pulses (kicks). Then stroboscopic observation of the system leads, at the classical level, to a symplectic map of $\mathscr{P}$ onto itself. Quantum mechanically, in the Schrödinger picture the one-step map is realized by a unitary operator U such that a state $|\psi\rangle$ of $\mathscr{H}$ is transformed, after a period T of the pulse, onto $|\psi\rangle^{(n+1)} = U|\psi\rangle^{(n)}$ [8]. Our quantum approach is different, and resembling much more the classical picture. From (3.8) we will construct a one-step map for the time evolution of the zeros, and study their phase space motion.

The first example deals with a system which, in spite of being periodically kicked, is classically integrable. The second has a regular to chaotic transition at the classical level.

4.1. *The linear case*

Consider a simple linear Hamiltonian on the generators

$$H = pS_z + \mu S_x \sum_{n=-\infty}^{\infty} \delta(t-n) \tag{4.1}$$

which can be interpreted physically as a spin s particle subjected to a constant magnetic field on the z direction plus a periodically ($T=1$) pulsed magnetic field applied in the x direction.

The classical equations of motion for the (normalized) spin vector $\boldsymbol{S} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} = \boldsymbol{S} \wedge \left(-\frac{\partial H}{\partial \boldsymbol{S}}\right) \tag{4.2}$$

integrated over a period $T=1$ give the following discrete map of the sphere onto itself

$$\boldsymbol{S}^{(n+1)} = R_x(\mu) R_z(p) \boldsymbol{S}^{(n)}. \tag{4.3}$$

Its physical interpretation is as follows: between two pulses, the constant magnetic field in the z direction produces a uniform precession of the spin by an angle p around the z axis. Then the magnetic field acts in the x direction, producing a sudden rotation of the spin by an angle μ around the x axis. The effective motion is just a rotation around a third axis $\boldsymbol{e}_r$ specified by the values of the parameters μ and p.

We now consider the quantum dynamics, i.e. a map for the zeros. Using (2.11), the Hamiltonian (4.1) gives, with $S_x = (S_+ + S_-)/2$,

$$\hat{H}(z, \partial_z) = \hbar p(z\partial_z - s) - \frac{\hbar\mu}{2}[(z^2-1)\partial_z + 2sz] \sum_{n=-\infty}^{\infty} \delta(t-n) \tag{4.4}$$

so that (3.8) become

$$\dot{z}_k = \mathrm{i}pz_k - \mathrm{i}\frac{\mu}{2}(z_k^2 - 1) \sum_{n=-\infty}^{\infty} \delta(t-n) \qquad k = 1, \ldots, N. \tag{4.5}$$

As mentioned before, due to the linearity of the Hamiltonian on the generators of the group we get a velocity $\dot{z}_k$ for the kth zero that depends on its position but *not* on the position of the other zeros.

These equations are easily integrated to obtain the one-step map. We denote $t^{(n)}$ the time just after the nth kick, and $t_+^{(n)}$ the time just before the $(n+1)$th kick. In the interval $t^{(n)} \leqslant t \leqslant t_+^{(n)}$ between two kicks, the position $z_k^{(n)}$ transforms to $z_{k+}^{(n)} = z_k^{(n)} \exp(\mathrm{i}p)$, i.e. the kth zero just rotates by p around the z axis, exactly as the classical spin. Then integrating (4.5) during the interval $t_+^{(n)} \leq t \leq t^{(n+1)}$ on which the kick acts yields the total map

$$z_k^{(n+1)} = \frac{(z_k^{(n)} \mathrm{e}^{\mathrm{i}p} + 1) + (z_k^{(n)} \mathrm{e}^{\mathrm{i}p} - 1)\mathrm{e}^{-\mathrm{i}\mu}}{(z_k^{(n)} \mathrm{e}^{\mathrm{i}p} + 1) - (z_k^{(n)} \mathrm{e}^{\mathrm{i}p} - 1)\mathrm{e}^{-\mathrm{i}\mu}} \qquad k = 1, \ldots, N \tag{4.6}$$

which is just, in stereographic projection, a rotation by μ around x of the kth zero from its position just before the kick $z = z_{k+}^{(n)}$. We have thus arrived at the conclusion that the motion of each *individual* zero coincides with the motion of a classical spin s particle having the same initial conditions. The quantum dynamics can then be viewed, in this simple case, as a collection of N non-intereacting point particles in phase space, all of them undergoing a classical motion [4, 9].

The stationary condition $z_k^{(n+1)} = z_k^{(n)}$ applied to (4.6) determines the two points arising from the intersection of the effective axis of rotation $\boldsymbol{e}_r$ with the sphere, for all k. These are obviously invariant under the dynamics. The zeros being indistinguishable, we have $N+1 = 2s+1$ different possibilities for the location of the N zeros on the two stable points.

The coincidence of the quantum motion of the zeros with the classical one is not just due to the integrability of the system, but to the linear dependence of H on the generators. For nonlinear integrable systems, they will be different. Analogously, in the latter case the stationary distributions of zeros will not generically be single points but will tend, in the semiclassical regime, to form lines with a typical distance of order $\hbar$ between the zeros [2].

4.2. *A quadratic kicked spin*

We now consider the Hamiltonian

$$H = \frac{p}{2} S_z^2 + \mu S_x \sum_{n=-\infty}^{\infty} \delta(t-n) \tag{4.7}$$

which has a nonlinear dependence on the generators of the group. This kind of peridoically kicked spin system has been considered by several authors [10, 11]. From (4.2) we now obtain the classical map

$$\boldsymbol{S}^{(n+1)} = R_x(\mu) R_z(p S_z^{(n)}) \boldsymbol{S}^{(n)} \tag{4.8}$$

whose structure is similar to the linear case (4.3). However, the rotation around the z axis is now done by an angle proportional to the z-component of the spin $\boldsymbol{S}$. At $\mu = 0$, the classical motion is integrable (a simple precession around the z axis with energy $E = p/2 \cos^2\theta$). As μ increases, the system undergoes a regular-to-chaotic transition. In figure 1 we show this transition for $p = 4\pi$ and $0 \leq \mu \leq 1$. The variables used to label $\mathcal{P}$ are $(\phi, \cos\theta)$, which are canonical conjugates.

Quantum mechanically the map is realized by the unitary transformation $|\psi\rangle^{(n+1)} = U|\psi\rangle^{(n)}$, $U = \exp(-\mathrm{i}\mu S_x/\hbar)\exp(-\mathrm{i}pS_z^2/(2\hbar))$. The corresponding equations of motion for the zeros, obtained from (3.8), are

$$\dot{z}_k = \mathrm{i}\hbar p\left[z_k^2 \sum_{j(\neq k)=1}^{N} \frac{1}{z_k - z_j} - (s - \tfrac{1}{2})z_k\right] - \mathrm{i}\frac{\mu}{2}(z_k^2 - 1)\sum_{n=-\infty}^{\infty}\delta(t-n)$$
$$k = 1, \ldots, N. \tag{4.9}$$

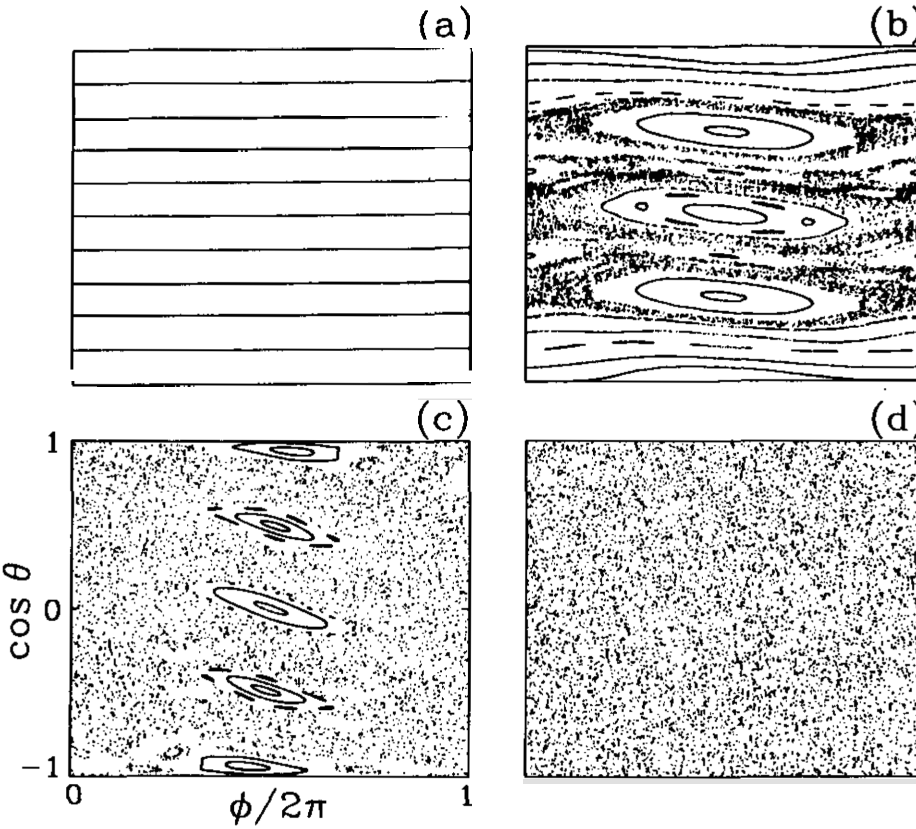

Figure 1. The classical map (4.8) for $p=4\pi$ and (a) $\mu=0$, (b) $\mu=0.1$, (c) $\mu=0.25$, (d) $\mu=1$.

The kick term is the same as in (4.1). However, the quadratic dependence of the Hamiltonian on S_z induces two-body interactions between the zeros, and their motion is now coupled.

As before, to obtain the one-step map we must integrate (4.9) between $t^{(n)} \leqslant t \leqslant t^{(n+1)}$. The motion between kicks

$$\dot{z}_k = \mathrm{i}\hbar p\left[z_k^2 \sum_{j(\neq k)=1}^{N} \frac{1}{z_k - z_j} - (s-\tfrac{1}{2})z_k \right] \qquad k=1,\ldots,N \tag{4.10}$$

corresponds classically to a uniform precession of the spin around the z axis provided by the term S_z^2 in (4.7). The quantum motion, described by (4.10), is much more complicated. We were not able to solve these equations analytically to obtain an explicit form for the map between two kicks, except for the simple case $s=1$ (two zeros), where we get for the position of the two zeros just before the $(n+1)$th kick

$$z_{1,2+}^{(n)} = (C^{(n)}\,\mathrm{e}^{\mathrm{i}p\hbar/2} \pm [(R^{(n)})^2 + (C^{(n)})^2(\mathrm{e}^{\mathrm{i}p\hbar}-1)]^{1/2})/2$$

$C^{(n)}$ and $R^{(n)}$ being the 'centre of mass' and 'relative' coordinates at time $t^{(n)}$, $C^{(n)} = z_2^{(n)} + z_1^{(n)}$ and $R^{(n)} = z_2^{(n)} - z_1^{(n)}$, respectively.

For an arbitrary $s>1$, (4.10) were solved numerically. We denote the change in the position of the kth zero during that period of time by $z_k^{(n)} \to z_{k+}^{(n)}$.

The kick is then easy to integrate. It is just a rotation of each zero around the x axis by an angle μ (cf (4.6)). The total map can thus be written

$$z_k^{(n+1)}=\frac{(z_{k+}^{(n)}+1)+(z_{k+}^{(n)}-1)\,e^{-i\mu}}{(z_{k+}^{(n)}+1)-(z_{k+}^{(n)}-1)\,e^{-i\mu}} \qquad k=1,\ldots,N. \tag{4.11}$$

Stationary configurations of zeros of this kicked spin system have been shown in [2] as a function of the parameter μ. We now show some global properties of the map (4.11) as the parameter μ changes from 0 to 1.

Since we want to emphasize the classical underlying structures, we have chosen large values of s and as initial configuration for the zeros a coherent-state wavepacket $|z_0\rangle$ peaked at some point z_0 of $\mathscr{P}$. The zeros are then concentrated at the opposite point (ϕ_*, θ_*) on the sphere. In practice, to avoid singularities in (4.10), we have placed the zeros over a small circle centred at (ϕ_*, θ_*). Other initial conditions, like a uniform distribution of zeros over $\mathscr{P}$, will mix several classically different structures, unless the system is completely chaotic. As in classical mechanics, we follow the trajectory of the zeros and we plot the successive positions $\{z_k^{(n)}\}$, $n=1,2,\ldots$. The result obtained in the asymptotic limit $t=n\to\infty$ gives an idea of the regions of phase space explored by the quantum state during its time evolution, and of the way this exploration is done. Although in a strict sense the distribution (2.10) cannot be interpreted as a phase space probability density since the coherent states $|z\rangle$ do not form an orthogonal set, in this 'complementary' picture of quantum mechanics regions of *high* density of zeros in $\mathscr{P}$ are regions where the system has a *low* 'probability' of being found there. This probabilistic interpretation becomes more and more correct as we approach the classical limit $s\to\infty$ where coherent states become orthogonal.

Figure 2(a) shows the regions of phase space explored by the zeros in the case $\mu=0$ and $2s=60$ up to a time $t=150$. In all the figures, the initial configuration of zeros is indicated by a star. The zeros describe in this case a very regular pattern, a sequence of horizontal lines (like the classical orbits). A vertical alignment also exists, but it occurs only for special values of p, such as 4π. This is also true for the small diagonal bands over which the zeros do not enter, as well as the vertical line at $\phi=\phi_*$. For other values of p the zeros tend to fill the horizontal lines uniformly. The total

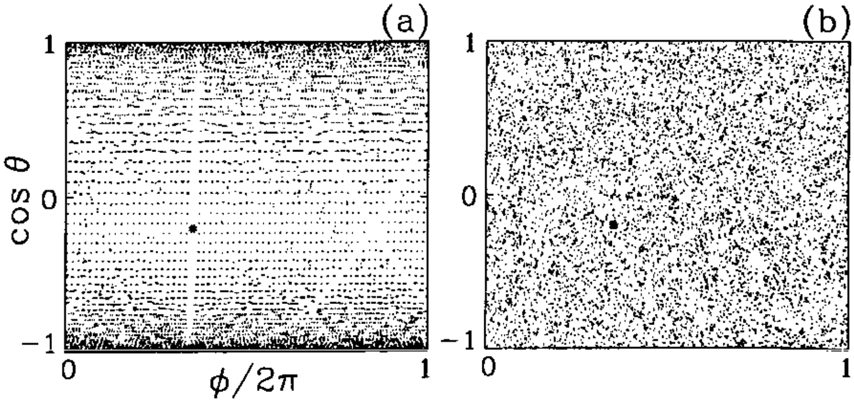

Figure 2. The quantum map (4.11) for $p=4\pi$, $N=2s=60$ and (a) $\mu=0$, (b) $\mu=1$. All the zeros are at $t=0$ concentrated on the star, defining thus as initial state a coherent-state wavepacket.

number of lines is of order $2s$, i.e. one line per zero. The same global structure as in figure 2(a) was observed for other initial points $(\phi_*, \cos\theta_*)$.

The motion of the zeros is completely different when the underlying classical dynamics is fully chaotic. In figure 2(b) we show the same plot as in figure 2(a), with the same initial condition, but for $\mu = 1$ (cf. figure 1(d)). Here the zeros tend to fill the whole phase space in an apparently disordered and uniform way, and the overall picture obtained quite resembles the classical one (the total number of points is the same in both figures). The question of the *ergodicity* of this distribution, i.e. whether or not the zeros in their time evolution cover the *whole* phase space is an important question which, however, cannot be answered by numerical methods. The latter is also true classically. For other positions of the initial coherent-state packet $|z_0\rangle$ the same pattern of figure 2(b) was observed, except when the wavepacket was concentrated over an unstable periodic orbit, like the period-one orbit located at $(\phi, \cos\theta) = (\pi, 0)$. In that case we have observed more structure in the covering of $\mathscr{P}$ by the zeros, and the appearance of certain regions where the zeros apparently do not enter.

Finally, figures 3(a) and (b) show the time evolution for two different initial conditions in the case of a mixed classical dynamics, $\mu = 0.25$. In the first case (figure 3(a)), we have concentrated the zeros at $(\phi_*, \cos\theta_*) = (0, 0)$ so that the wavepacket is peaked on the regular region surrounding $(\phi, \cos\theta) = (\pi, 0)$. The fact that the zeros in their time evolution do not enter that classically regular region indicates that the quantum state remains concentrated there, and does not spread all over $\mathscr{P}$. However, the motion of the zeros does not present a regular pattern, as in figure 2(a). The appearance of regions of high density of zeros (like those close to the north and south poles) suggests simple coherent effects in the time propagation of the quantum state. On the other hand, figure 3(b) corresponds to a wavepacket initially centred on a chaotic region of $\mathscr{P}$. Now the zeros tend to cover the whole phase space. This is not done in a uniform way, and a lot of structure is observed. The regions of high density of zeros observed where regular invariant tori exist (cf figure 1(c)) are easy to understand, since the system has a low probability of entering that region, and remains wrapped around in the chaotic domain. However, the zeros also tend to cover the chaotic regions of $\mathscr{P}$ where the wavefuntion spends most of its time. This makes a

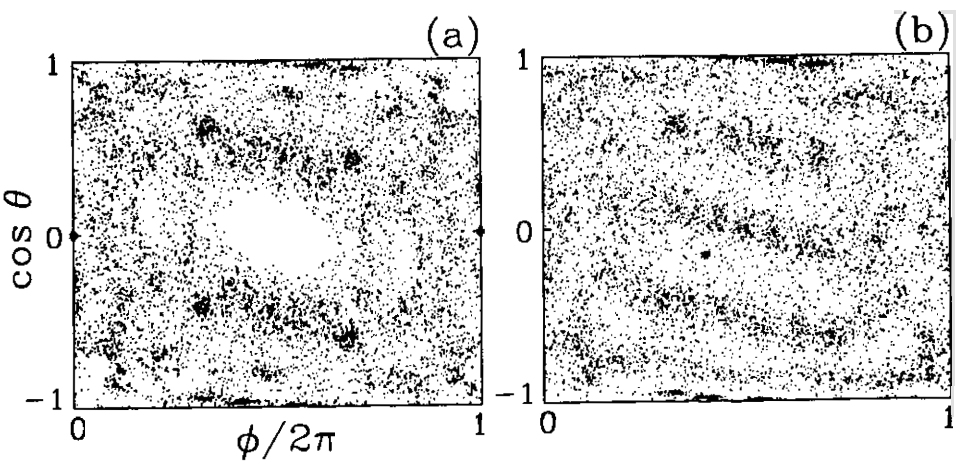

Figure 3. The same as in figure 2 but for $\mu = 0.25$ and two different positions (a) and (b) for the initial wavepacket.

clear difference between states of mixed systems propagating in chaotic regions of phase space, like the present one, and those propagating in regular regions, like that of figure 3(*a*).

5. Conclusion

(i) The covering of phase space by the zeros as some boundary conditions are changed was recently exploited to interpret the localization properties of eigenstates of quantized classically chaotic maps [12]. In that context, zeros of delocalized states (having a Chern index different from zero) were shown to cover the whole phase space. Semiclassically, those states are associated with chaotic regions of phase space. In contrast, zeros of eigenstates associated with regular regions (localized states having zero Chern index) do not cover the entire phase space. The present study seems to confirm, but now on dynamical grounds, such a scheme.

(ii) We have chosen the case of the SU(2) group for technical simplicity, but other groups can be treated in the same way. Equation (3.3) is general, but the factorization formula (2.7) and the differential representation of generators are obviously not, so that the final form of the equations of motion will depend on the group.

(iii) In the semiclassical limit, and at least for sufficiently short times, the phase space location of zeros at time $t=n$ can be interpreted as the interference of different branches of the classical action (a sort of anti-Stokes line, see [2] for details). When the initial state is, as in section 4.2, a coherent-state wavepacket $|z_0\rangle$, what we have studied are just the zeros of the propagator in the coherent-state representation $\langle z|U^n|z_0\rangle$ as a function of time $t=n$, since $|\psi\rangle^{(n)}=U^n|z_0\rangle \Rightarrow \psi(z, t=n)=\langle z|U^n|z_0\rangle$. Adachi [13] has interpreted the location of zeros of $\langle z|U^n|z_0\rangle$ in terms of 'folding' of wavefunctions in phase space, the basic mechanism, together with 'stretching', generating a chaotic dynamics at the classical level. More recently D'Ariano *et al* [14] have also studied, in a kicked spin model, the short-time propagation of zeros of initial coherent-state wave packets centred at unstable periodic orbits.

(iv) For multidimensional systems, the basic structure (2.7) is lost since we now run into the theory of analytic functions of several variables. I sketch here a possible way to overcome the problem by considering a quantum analogue of the classical Poincaré section construction. Consider for definiteness a system of two coupled spins s_1 and s_2. Then the analytic coherent-state representation $|z_1 z_2\rangle=|z_1\rangle \otimes |z_2\rangle$ of any vector of Hilbert space

$$\psi(z_1, z_2)=\langle z_1 z_2|\psi\rangle=\sum_{m_1=-s_1}^{s_1}\sum_{m_2=-s_2}^{s_2} c_{m_1 m_2} z_1^{s_1+m_1} z_2^{s_2+m_2} \tag{5.1}$$

is an analytic function of two complex variables. Instead of the full function $\psi(z_1, z_2)$ defined on a 4D space, let us consider

$$\begin{aligned} S_{m_2}(z_1) &= \sum_{m_1=-s_1}^{s_1} c_{m_1 m_2} z_1^{s_1+m_1} \qquad m_2=-s_2, \ldots, s_2 \\ &= \mathcal{N}_1 \prod_{k=1}^{2s_1} (z_1 - z_{1k}). \end{aligned} \tag{5.2}$$

There are $(2s_2+1)$ of such functions labelled by the index m_2, and each of them has $2s_1$ zeros on the plane spanned by z_1. The set of $(2s_2+1)\times 2s_1$ zeros clearly completely determines (up to proportionality) the quantum state. Moreover, $S_{m_2}(z_1)$ can be considered as a quantum 'Poincaré section' of the 4D $\psi(z_1, z_2)$ through the section plane $m_2 = \text{constant}$. The classical analogue of this procedure is to fix the projection of the spin s_2 over the vertical axis, i.e. the plane $\cos\theta_2 = m_2/s_2 = \text{constant}$. ϕ_2 being determined by energy conservation, we get a surface of section in the variables $((\phi_1, \theta_1)$, to be compared with $S_{m_2}(z_1)$. Equation (5.2) is the coherent-state representation of a 1D system (cf (2.6)). However, an eventual transition to chaos will not be produced here via a time dependence on H but through the coupling to the other spin.

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